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Complex solitons with power law behaviour in Bose-Einstein condensates near Feshbach resonance

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Complex, localized stable solitons, characterized by a power law behaviour, are found for a quasi-one-dimensional Bose-Einstein condensate near Feshbach resonance. Both dark and bright solitons can be excited in the experimentally allowed parameter domain, when two and three-body interactions are respectively repulsive and attractive. These solutions are obtained for non-zero chemical potential, unlike their unstable real counterparts which exist in the limit of vanishing μ. The dark solitons travel with constant speed, which is quite different from the Lieb mode, where profiles with different speeds, bounded above by sound velocity can exist for specified interaction strengths.

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The dynamics of non-linear waves in Bose-Einstein condensate (BEC) is a subject of immense theoretical and experimental interest in current literature. The recent observation of dark [1], bright solitons [2, 3, 4, 5], soliton trains [6] and Faraday waves [7] have given considerable impetuous to the investigation of the formation mechanism and control of various non-linear excitations in the quasi-1D scenario [8]. The mean field equation, governing the dynamics of BEC, is the non-linear Schrödinger equation with a harmonic trap. The non-linearity originating from the two-body interaction is characterized by the s-wave scattering length 'a', which can be controlled through Feshbach resonance [9], as also the width of the transverse profile [7]. For $a > 0$, elastic interaction is repulsive and the BEC is stable. Negative scattering length implies attractive interaction, where the condensate is found to be stable up-to a certain limit of the number of atoms [10, 11, 12].

The three-body interaction can be generally treated as a perturbation over the two-body case; it becomes significant for short range and larger scattering length, as is the case near Feshbach resonance. For a dense atomic media also, three-body interaction plays an important role. It is estimated for Rb-BEC that, the real part of the three-body interaction term is $10^3 - 10^4$ times larger than the imaginary part [13, 14, 15, 16]. Hence, we do not consider the three-body recombination here, when the corresponding coupling constant is imaginary. A number of theoretical studies have been carried out considering three body interaction in both three- and quasi-one-dimensions [17, 18, 19, 20, 21, 22]. Localized soliton solutions of both elliptic function and power-law type have also been investigated [23, 24, 25]. Dark soliton of secant hyperbolic form manifested in purely repulsive three-body interaction regime [24], relevant for Tonks-Girardeau gas [26, 27, 28]. In this case, the soliton velocity is bounded above by sound velocity. Real solitons of both types were also analyzed in [25], where the algebraic one was found to exist only in the $\mu \to 0$ limit and was unstable.

In this letter, we demonstrate the existence of power-law type complex solitons in the presence of repulsive two- and attractive three-body interactions. Unlike the real case, the obtained dark and bright soliton solutions can exist for non-vanishing μ and are stable. The dark solitons have a constant velocity determined by the interaction strengths, which is quite different from the Lieb mode case [24]. Their profiles can change as a function of the parameters of the theory. The corresponding velocities change from zero to sound velocity. Interestingly, in the parameter domain where soliton velocity equals sound velocity, it is found that the Bogoliubov dispersion is of the quadratic type. For specificity, we consider ^{87}Rb with $m = 1.44 \times 10^{-25}$ Kg and the axial density σ_0 in the range $5.43 \times 10^7 \text{cm}^{-1} - 9.67 \times 10^8 \text{cm}^{-1}$. The transverse trap-frequency is taken as $\omega_\perp = 2\pi \times 140$ rad/sec and the two-body coupling constant $g_2 = 4.95\hbar \times 10^{-11} \text{cm}^3$/sec. The three-body interaction coefficient g_3 has already been estimated [16, 29, 30, 31, 32]. The above parameters allow the present solitons to exist, in the domain of g_3, taking values from -10^{-27}cm^6/sec to -10^{-26}cm^6/sec (scaled by $\hbar$). This is in the range of theoretically predicted value for ^{87}Rb.

A linear stability analysis using spectral method is carried out, which shows that the obtained solutions are stable against small perturbations in both dark and bright soliton regimes. Modulational instability (MI) analysis [33, 34, 35] reveals that the parameter regimes relevant for the solutions are away from the domain of instability.

The 3D Gross-Pitaevskii (GP) equation for the wave function $\Psi(r, t)$, with an additional three-body interaction, is given by

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + \left(V + g_2 \, |\Psi|^2 + g_3 |\Psi|^4 - \mu\right)\Psi, \quad (1)$$

where μ is the chemical potential. The cylindrical harmonic trap is given by $V = m\omega_\perp^2(x^2 + y^2)/2$ with a tight

transverse confinement. For sufficiently small transverse dimension of the cloud, the wave function can be written as $\psi(r,t) = f(z,t)\,\phi_0$ with $\phi_0 = \sqrt{\frac{1}{\pi a_\perp^2}}\exp(-\frac{x^2+y^2}{2a_\perp^2})$ and $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$. The longitudinal envelope function $f(z,t)$ obeys [36, 37],

$$i\hbar\frac{\partial f}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 f}{\partial z^2} + \left(\tilde{g_2}|f|^2 + \tilde{g_3}|f|^4 - \mu\right)f, \quad (2)$$

where the reduced interaction coefficients are

$$\tilde{g_2} = \frac{m\omega_\perp}{2\pi\hbar}\,g_2, \quad \tilde{g_3} = \frac{m^2\omega_\perp^2}{3\pi^2\hbar^2}\,g_3. \quad (3)$$

For the space-time independent solution, chemical potential can be written in terms of the asymptotic density σ_0;

$$\mu = (\tilde{g_2} + \tilde{g_3}\sigma_0)\sigma_0. \quad (4)$$

The superfluid velocity is obtained from the continuity equation:

$$v = u(1 - \frac{\sigma_0}{\sigma}), \quad (5)$$

where $v = \frac{\hbar}{m}\frac{\partial\theta}{\partial\xi}$ and $f = \sqrt{\sigma(\xi)}\,e^{i\theta(\xi)}$. The hydrodynamic equation for the density is then,

$$\begin{aligned} -\frac{\hbar^2}{2m}(\sigma_z^2 - 2\sigma\sigma_{zz}) &= 4\tilde{g_3}\sigma^4 + 4\tilde{g_2}\sigma^3 \\ &- (4\mu + 2mu^2)\sigma^2 + 2mu^2\sigma_0^2. \quad (6) \end{aligned}$$

A power law ansatz

$$\sigma(\xi) = \sigma_0\left(1 - \frac{B}{1 + D\xi^2}\right), \quad (7)$$

is found to solve Eq. 6, where B and D are given by,

$$B = \frac{3\tilde{g_2} + 8\tilde{g_3}\sigma_0}{2\tilde{g_3}\sigma_0}, \quad D = -\frac{m}{\hbar^2}\frac{(3\tilde{g_2} + 8\tilde{g_3}\sigma_0)^2}{6\tilde{g_3}},$$
$$\text{with} \quad u = \pm\Big(\frac{\tilde{g_2}\sigma_0 + 2\tilde{g_3}\sigma_0^2}{m}\Big)^{\frac{1}{2}}. \quad (8)$$

It is transparent that, non-singular solutions exist only when $\tilde{g_3}$ is negative, i.e., attractive three-body interaction. The value of $\tilde{g_2}$ should be positive from the reality of the soliton velocity, implying repulsive two-body interaction. As mentioned before, u is a constant for given parameter values and density. This situation is quite different from the Lieb-mode case, where the soliton velocity can take different values, bounded above by the sound velocity. The obtained solutions can be categorized into three different classes depending on the values of $\tilde{g_3}$ for a given $\tilde{g_2}$ and σ_0: (i) A dark solition in the range $-\tilde{g_2}/2\sigma_0 \le \tilde{g_3} < -3\tilde{g_2}/8\sigma_0$, (ii) a constant background for $\tilde{g_3} = -3\tilde{g_2}/8\sigma_0$ and (iii) a bright soliton for $-3\tilde{g_2}/8\sigma_0 \le \tilde{g_3} < -0.28\tilde{g_2}/\sigma_0$. In these regimes μ is a real positive quantity. For $\mu = 0$, one only obtains a real soliton [25]. Figure 1 shows the density profiles of dark and bright solitons for different values of $\tilde{g_3}$. Usually, repulsive interaction alone creates dark soliton, whereas attractive one results in bright solitons in BEC. As both types of forces are present in the present system, one gets dark and bright solitons, depending on the values of the coupling constants $\tilde{g_2}$ and $\tilde{g_3}$. In Fig. 1, $\tilde{g_3}$ is increased from dark to bright soliton for a particular value of $\tilde{g_2}$. The density profile smoothly transits from dark soliton to the bright one. Hence, larger the value of three-body interaction, greater is the accumulation of atoms in the condensate. Physically it amounts to increasing the local density of atoms for going from dark to bright regime. This leads to a depletion of atoms in the background. The solid line in Fig. 1 corresponds to $u = 0$ case. Thick solid line is the homogeneous background $\sigma = \sigma_0$, where $u = \pm\frac{1}{2}\sqrt{g_2\sigma_0/m}$.

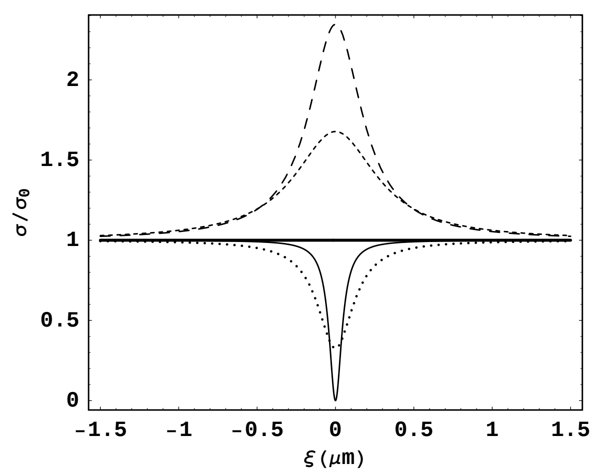

FIG. 1: The density profiles of soliton solutions for different three-body interactions with $g_2 = 4.95\hbar \times 10^{-11}\text{cm}^3/\text{sec}$. The obtained dark solitions for $\tilde{g_3} = -\tilde{g_2}/2\sigma_0$ (solid line), $\tilde{g_3} = -0.45\tilde{g_2}/\sigma_0$ (dotted line) and bright solitons for $\tilde{g_3} = -0.32\tilde{g_2}/\sigma_0$ (small-dashed line), $\tilde{g_3} = -0.28\tilde{g_2}/\sigma_0$ (long-dashed line). The thick solid line represents the constant background density σ_0 for $\tilde{g_3} = -3\tilde{g_2}/8\sigma_0$.

The appropriately normalized energy functional

$$E = \int \Big[\frac{\hbar^2}{2\,m}\;\frac{\partial f}{\partial z}\frac{\partial f^*}{\partial z} + \frac{\tilde{g_2}}{2}(ff^*)^2 - \frac{\tilde{g_2}}{2}\sigma_0^2$$
$$+\frac{\tilde{g_3}}{3}(ff^*)^3 - \frac{\tilde{g_3}}{3}\sigma_0^3 - \mu(ff^* - \sigma_0)\Big]dz,$$

yields

$$E = \sqrt{\frac{3\pi^2\hbar^2}{2\,m}}\;\frac{\tilde{g_2}\,|3\tilde{g_2} + 8\tilde{g_3}\sigma_0|}{16|\tilde{g_3}|^{3/2}}, \quad (9)$$

for the soliton profile. Dark soliton with $u = 0$ corresponds to energy $E = \pi\hbar\sigma_0\sqrt{3\tilde{g_2}\sigma_0/(64m)}$, whereas it

goes to zero when the background is uniform. Momentum of the condensate profile

$$P = \frac{-i\hbar}{2}\int dz[f^* f_z - f_z^* f] = m\int dz(\sigma - \sigma_0)v(z),$$

gives

$$P = \pi\hbar\sigma_0 \frac{u}{|u|}\Big(1 - \sqrt{\frac{3(\tilde{g}_2 + 2\tilde{g}_3\sigma_0)}{-2\tilde{g}_3\sigma_0}}\,\Big). \tag{10}$$

It reaches maximum value ($P_{\max} = \pi\hbar\sigma_0$) when $\tilde{g}_3 = -\tilde{g}_2/(2\sigma_0)$. Figure 2 depicts the variation of energy with momentum for different three-body interaction strengths. Positive momentum is the region of dark soliton, where as negative one corresponds to bright soliton. Energy and momentum vanish at the transition point $\sigma = \sigma_0$. Dispersion graph is stiffer in the bright soliton regimes.

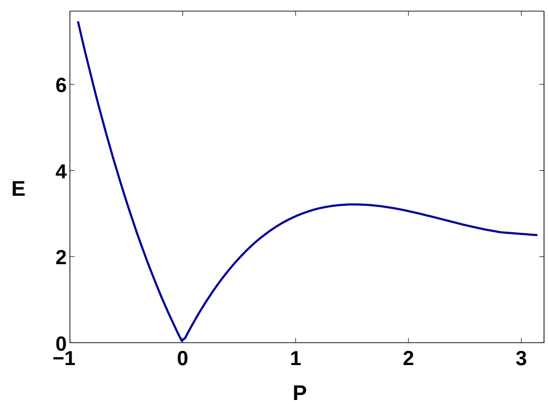

FIG. 2: Energy vs momentum for the dark and bright solitons for $-\tilde{g}_2/2\sigma_0 \leq \tilde{g}_3 \leq -0.32\tilde{g}_2/\sigma_0$ with the same g_2 used in Fig. 1. Energy and momentum are respectively scaled by $\hbar^2{\sigma_0}^2/m \times 10^{-4}$ and $\hbar\sigma_0$.

The number of atoms in the condensate, normalized to vanish at $\sigma = \sigma_0$,

$$\begin{aligned} N &= \int dz(\sigma_0 - \sigma(z)) \\ &= \left(\frac{3\pi^2\hbar^2\sigma_0^2}{m|\tilde{g}_3|}\right)^{1/2} |3\tilde{g}_2 + 8\tilde{g}_3\sigma_0|, \end{aligned} \tag{11}$$

shows that the maximum deficiency of atoms in the dark soliton regime is $N = (6\pi\hbar^2\sigma_0^3\tilde{g}_2/m)^{1/2}$.

We now analyze the dynamical stability of obtained solutions using the spectral method [38, 39]. A small perturbation $e^{\lambda\, t}\phi(\xi)$ of soliton solution satisfies

$$\mathbf{A}\,\vec{\varphi} = \lambda\,\mathbf{J}\,\vec{\varphi}, \tag{12}$$

where $\vec{\varphi}$ is a two-dimensional vector and its components are real and imaginary parts of the perturbation: $\vec{\varphi} = (\phi_1\ \phi_2)^T$. Here, $\mathbf{J}$ is a two-dimensional matrix with $J_{11} = J_{22} = 0$ and $J_{12} = -J_{21} = 1$. The elements of the matrix operator $\mathbf{A}$ are

$$\begin{aligned} A_{11} &= \frac{\hbar^2}{2\,m}\frac{\partial^2}{\partial\xi^2} - \tilde{g}_2(3\,f_1^2 + f_2^2) - \tilde{g}_3(5\,f_1^4 + f_2^4 + 6\,f_1^2\,f_2^2) + \mu, \\ A_{12} &= \hbar\,u\frac{\partial}{\partial\xi} - 2\,\tilde{g}_2\,f_1\,f_2 - 4\,\tilde{g}_3\,f_1\,f_2\,|f|^2, \\ A_{21} &= -\hbar\,u\frac{\partial}{\partial\xi} - 2\,\tilde{g}_2\,f_1\,f_2 - 4\,\tilde{g}_3\,f_1\,f_2\,|f|^2 \quad \text{and} \\ A_{22} &= \frac{\hbar^2}{2\,m}\frac{\partial^2}{\partial\xi^2} - \tilde{g}_2(f_1^2 + 3\,f_2^2) - \tilde{g}_3(f_1^4 + 5\,f_2^4 + 6\,f_1^2\,f_2^2) + \mu, \end{aligned}$$

where $f = (f_1 + i\,f_2)$. The soliton solution is stable if real part of the eigenvalue λ is negative. ϕ_1 and ϕ_2 are expanded into a spectral series over 800 modes. This numerical analysis shows that both bright and dark solitons solutions are stable in the entire domain of the solutions.

It is now worth investigating the issue of modulation instability since the three-body interaction is attaractive. Phenomenon of modulational instability has been extensively investigated in literature for BEC [40, 41, 42]. A single component BEC with an attractive atom-atom interaction, can result in modulational instability, when the density of atoms exceeds a certain critical value. We assume $f = (f_0 + \tilde{f})exp(i\tilde{\phi})$, where the infinitesimal fluctuation $\tilde{f}$ is given by

$$\tilde{f} = \tilde{f}_1 cos(Kz - \Omega t) + i\tilde{f}_2 sin(Kz - \Omega t). \tag{13}$$

Ω and K are respectively, the frequency and propagation constant, of the modulated wave. The above transformation produces two sets of equations involving $\tilde{f}_1$ and $\tilde{f}_2$. Non trivial solutions are obtained only if K and Ω satisfy the dispersion relation, $2m\Omega^2 = K^2(\hbar^2K^2/2m - 4|\tilde{g}_3|{f_0}^4 + 2\tilde{g}_2{f_0}^2)$, where $\tilde{g}_3 < 0$ and $\tilde{g}_2 > 0$. If $\hbar^2K^2/2m < (4|\tilde{g}_3|{f_0}^4 - 2\tilde{g}_2{f_0}^2)$, it would show modulation instability. This condition immediately implies $\tilde{g}_2 < 2|\tilde{g}_3|\sigma_0$, which is not in the allowed parameter range for the obtained solutions. Thus our solutions are modulationally stable.

In conclusion, complex soliton solutions with power law decay have been identified in the quasi-one-dimensional GP equation with repulsive two- and attractive three-body interactions. These solutions, when superfluid velocity depends on density, show a slower asymptotic decay compared to the elliptic function type solutions. We have considered the parameters relevant to ^{87}Rb, with the three-body coupling constant in the theoretically predicted range. This opens the possibility of observing these complex solitons in realistic BEC. Soliton velocity is fixed by the strength of the interactions and are stable against small perturbations. They are also modulationally stable. One would like to study their behaviour in a trap for the purpose of coherent control. The analysis of two-soliton sector is also an interesting problem, as is the investigation in higher dimensions.

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